we have that

$$\mathbf{x}_2 = -\omega^2(\mathbf{T}_{12}{}^T\mathbf{m}_{11}\mathbf{T}_{12})^{-1}\mathbf{T}_{12}{}^T\mathbf{m}_{11}\mathbf{k}_{11}{}^{-1}\mathbf{m}_{11}\mathbf{x}_1 \quad (12)$$

Finally, substituting Eq. (12) into (11) and dividing by ω^2 we obtain the standard equation

$$(1/\omega^2)\mathbf{x}_1 = (\mathbf{I} - \mathbf{T}_{12}(\mathbf{T}_{12}{}^T\mathbf{m}_{11}\mathbf{T}_{12})^{-1}\mathbf{T}_{12}{}^T\mathbf{m}_{11})\mathbf{k}_{11}{}^{-1}\mathbf{m}_{11}\mathbf{x}_1 \quad (13)$$

Assuming that $\mathbf{k}_{11}{}^{-1}$ is available, the only matrix inversion required in Eq. (13) is for the product $(\mathbf{T}_{12}{}^T\mathbf{m}_{11}\mathbf{T}_{12})$ which, for three-dimensional structures, is of the order (6×6). This compares with two inversions in Eq. (9), one for $(\mathbf{T}_{12}{}^T\mathbf{m}_{12} + \mathbf{m}_{22})$ of order (6×6) in $\mathbf{R}$ and another for $(\mathbf{I} - \mathbf{T}_{12}\mathbf{R})$ of order $(n \times n)$, where n is the number of degrees of freedom in $\mathbf{x}_1$. It follows therefore that, in general, if the introduction of a few massless node points in the structure is an acceptable idealization so that $\mathbf{m}_{22} = \mathbf{0}$ and $\mathbf{m}_{12} = \mathbf{0}$ (zero inertia in the direction of $\mathbf{x}_2$) then it would be preferable to compute the nonzero frequencies of a free-free system from Eq. (13).

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Reply by Authors to L. Meirovitch and J. S. Przemieniecki

MALCOLM NEWMAN*
Analytical Mechanics Associates Inc., Jericho, N. Y.
AND
I. U. OJALVO†
Grumman Aerospace Corporation, Bethpage, N. Y.

THE approach presented in Meirovitch's excellent book[1] was originally developed by Mack[2] as cited in our Note. In essence, this method reduces the semidefinite free-free system to a positive definite eigenvalue problem by measuring elastic deformations of the structure from a rigid-body reference position. The singular stiffness matrix is reduced to a positive definite form by eliminating rows and columns corresponding to the rigid-body degrees of freedom and a reduced mass matrix may be obtained by either requiring that the system moments be zero or, equivalently, by imposing orthogonality of the flexible modes with the rigid-body mode shapes. However, as was stated in our note,[3] the eigenvectors of the reduced problem obtained in this manner constitute relative coordinates with respect to the assumed reference position. If the modes are required in inertial coordinates, which is usually the case in aeroelastic dynamic stability studies, an additional coordinate transformation must be performed.

Our Note circumvents this extra computation by employing a simple rigid-body transformation matrix to reduce the free-free system's stiffness and mass matrices in such a way as to obviate the need for employing relative coordinates. Thus, the flexible vibration modes are obtained directly in inertial coordinates as the eigenvectors of the reduced system.

The advantages of a symmetrical formulation, as noted in Ref. 1, are not disputed and, for this reason, were presented as Eq. (17) of our Note. We leave it to the reader to decide for himself whether "virtually the entire material presented in the Note" can be found in Ref. 1.

Turning to Przemieniecki's comments on the relative value of obtaining a nonsingular stiffness matrix, the authors agree that eigenvalue algorithms are available to accommodate singular matrices. However, their use often involves penalties with regard to accuracy or computation time. It should also be noted that many highly reliable and efficient numerical methods require the use of positive definite matrices. This fact, coupled with the increasing popularity of the stiffness method[4] (which does not deliver $k_{11}{}^{-1}$ automatically) over the force method, seems to make our formulation a useful alternative to the existing ones.

Another issue introduced by Przemieniecki, following his Eq. (10), concerns the possibility of a singular mass matrix. As implied in our note, we did not consider this situation as it was assumed that m_{11} and m_{22} were positive definite. However, since this possibility exists and the point has been raised, we would suggest elimination of the noninertial degrees of freedom prior to elimination of the rigid-body modes. A standard procedure for accomplishing this is described in Ref. 5.

In connection with this latter point, it should be noted that the discussor's Eq. (13) still contains all the rigid-body modes (the matrix in braces on the right-hand side of the equation is singular). Thus, as noted earlier, Eq. (13) precludes the use of eigenvalue computational methods which require positive definite matrices.

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* Manager, Structural Mechanics.

† Structural Mechanics Staff Engineer.

Comment on "The Eigenvalue Problem for Structural Systems with Statistical Properties"

LARRY A. KIEFLING*
NASA Marshall Space Flight Center, Ala.

Nomenclature

K = stiffness matrix
M = mass matrix
λ_i = ith eigenvalue

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* Chief, Structural Dynamics Section, Aero-Astrodynamics Laboratory.

X_i = ith eigenvector
dX_i = perturbation of the ith eigenvector
F_i = $K - \lambda_i M$
A_i = $[-dK + d\lambda_i M + \lambda_i dM]\{X_i\}$
n = size of F_i matrix
$d\bar{X}_i$ = dX_i with element "s" removed
$\bar{F}_i$ = F_i with column "s" and row "t" removed
$\bar{A}_i$ = A_i with element "t" removed
$\bar{M}_i$ = $X_{iT} dM X_i$

COLLINS and Thomson[1] calculate the variation of the eigenvalues of $d\lambda_i$ and the eigenvectors dX_i for the problem $KX_i = \lambda_i M_i X_i$. Although the given equation for the variation of the eigenvector dX_i is correct, it requires the prior calculation of all the other eigenvectors of the system. This is not difficult for small systems, but certainly is a problem when larger problems are considered. Also, since each eigenvector is linearly independent of all other eigenvectors, one would expect its perturbations to also be independent. This will now be shown to be true.

Fox and Kapoor[2] have, in fact, recently published equations to calculate the perturbations of the eigenvector without using the other eigenvectors. They report numerical difficulties, however. Simplification of their method, which will reduce the number of computations, should reduce roundoff error, and eliminate the near-singular condition which exists in their system, is possible.

The basic eigenvalue equation is

$$[K - \lambda_i M]\{X_i\} = 0 \tag{1}$$

Differentiating with each term considered as a variable produces

$$[K - \lambda_i M]\{dX_i\} = [-dK + d\lambda_i M + \lambda_i dM]\{X_i\} \tag{2}$$

The value $d\lambda_i$ may be calculated as Collins and Thomson state, and Eq. (2), for convenience, may be reduced to

$$[F_i]\{dX_i\} = \{A_i\} \tag{3}$$

If F_i is an $n \times n$ matrix and λ_i is a single root, then the rank of F_i is known to be $n - 1$. The vector dX_i then has no unique solution, but if a value is assumed for one element of dX_i, the others can be uniquely determined. If the original set of equations represented by matrix Eq. (3) is completely coupled, as is common for structures problems, each equation is consistent with the other equations. Therefore, any equation can be discarded. If some of the original equations are completely uncoupled, for example,

$$\left[\begin{array}{c|c} F_{1i} & 0 \\ \hline 0 & F_{2i} \end{array}\right] \left\{\begin{array}{c} dX_{1i} \\ \hline dX_{2i} \end{array}\right\} = \left\{\begin{array}{c} A_{1i} \\ A_{2i} \end{array}\right\} \tag{3a}$$

then X_i will have the form

$$\left\{\begin{array}{c} X_{1i} \\ \hline 0 \end{array}\right\}$$

for a root of F_1 and

$$\left\{\begin{array}{c} 0 \\ \hline X_{2i} \end{array}\right\}$$

for a root of F_2. For a root of F_1, X_{2i} is always zero and hence dX_{2i} will always be zero by definition. Hence the element to be arbitrarily set zero must be a part of dX_{1i} and the equation to be discarded also selected from F_{1i}. The process is analogous to the fixing of a free-free stiffness matrix performed by structures analysts.

The obvious approach is to set one element of dX_i to zero, equivalent to normalizing the perturbed eigenvector equal to the unperturbed eigenvector at that element. The result is a consistent set of n equations in $n - 1$ unknowns.

In matrix notation the solution is

$$\{d_i\bar{X}\} = [\bar{F}_i]^{-1}\{\bar{A}_i\} \tag{4}$$

If it is desired to keep a constant generalized mass, Eq. (15) of Ref. 2 may be used. In our nomenclature it is

$$\left[\begin{array}{c} F_i \\ \hline 2X_{iT}[M] \end{array}\right]\{dX_i\} = \left\{\begin{array}{c} A_i \\ \hline \bar{M}_i \end{array}\right\} \tag{5}$$

Elimination of any row of F_i and the corresponding element of A_i, consistent with the coupling criteria discussed earlier, will yield a solvable set of equations. The process given involves fewer calculations and should give less roundoff than the more general solution given by Fox and Kapoor.

An alternative method of keeping a constant generalized mass is to first perform the solution of Eq. (4) then renormalize. Denoting the renormalized variation of the eigenvector by dX_i', the solution is

$$\{dX_i'\} = \left(\frac{\{X_i\}_T[M]\{X_i\}}{\{X_i + dX_i\}_T[M]\{X_i + dX_i\}}\right)^{1/2} \times \{X_i + dX_i\} - \{X_i\} \tag{6}$$

The advantage of solving Eqs. (4) and (6) over solving Eq. (5) is that elimination of the same row and column from the F_i matrix produces a symmetric matrix to be inverted. This should permit considerable decrease in computer time.

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Comment on "Angle-of-Attack Convergence and Windward-Meridian Rotation Rate of Rolling Re-Entry Vehicles"

CHARLES H. MURPHY*
Ballistic Research Laboratories, Aberdeen Proving Ground, Md.

Nomenclature

$C_{N\alpha}$ = aerodynamic normal force derivative
C_{mq} = aerodynamic pitch damping derivative
d = aerodynamic reference diameter
I = pitch or yaw moment of inertia
I_x = roll moment of inertia
l = static margin (distance of center of pressure aft of center of mass)
m = vehicle mass
M_η = aerodynamic pitch moment
M_ζ = aerodynamic yaw moment
p = roll rate
q = dynamic pressure
S = aerodynamic reference area
u = vehicle velocity
$\hat{v}$ = velocity component along Y axis

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* Chief, Free Flight Aerodynamics Branch. Associate Fellow AIAA.